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Canonical structure of topologically massive gravity with a cosmological constant

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Abstract: We study the canonical structure of three-dimensional topologically massive gravity with a cosmological constant, using the full power of Dirac's method for constrained Hamiltonian systems. It is found that the dimension of the physical phase space is two per spacetime point, which corresponds to a single Lagrangian degree of freedom. The analysis of the AdS asymptotic region reveals a remarkable relation to 3D gravity with torsion: in the limit of vanishing torsion, the conserved charges and asymptotic symmetries of the two theories become identical.

Keywords: Classical Theories of Gravity, Space-Time Symmetries

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1 Introduction

Three-dimensional (3D) gravity, with or without a cosmological constant Λ, is a topological theory, in which there are no local physical degrees of freedom [1]. An interesting modification of 3D gravity is obtained by adding the gravitational Chern-Simons term. General relativity with a Chern-Simons term is known as topologically massive gravity (TMG), and in contrast to pure general relativity, it is a dynamical theory with a local propagating degree of freedom, the massive graviton [2]. More generally, having in mind a rich dynamical structure found in general relativity with a cosmological constant [3], one expects that its extension by the gravitational Chern-Simons term, denoted shortly as TMG_Λ, may provide a new insight into the black hole dynamics and the asymptotic structure of spacetime [4].

Both the gauge structure of a dynamical system and its physical content are most clearly understood in the canonical formalism. The constrained Hamiltonian analysis of

the full TMG_Λ was carried out recently in [5–7] (for the case $\Lambda = 0$, see [8]). The treatment of the problem is characterized with complicated calculational details, which might be a reason for significant inconsistencies in the conclusions. Namely, Park [5] found that the number of degrees of freedom in configuration space is $N_c = 3$ (one "for each internal index"), Carlip [6] obtained $N_c = 1$, while Grumiller et al. [7] also found $N_c = 1$, but in the chiral version of the theory [9].

Our original motivation for studying TMG_Λ was to understand the relation between 3D gravity and 3D gravity with torsion [10, 11], and explore the influence of geometry on the gravitational dynamics. After reading the literature, we learned that the constraint structure of TMG_Λ has a rather controversial status [5–7], and we focused our attention on this issue. Our present study of the structure of TMG_Λ is based on using the full power of Dirac's canonical formalism [12], and it leads to the conclusion $N_c = 1$. The consistency of our results is checked by comparing with the Lagrangian formalism, and by constructing the canonical gauge generator. As a byproduct of our analysis, we are now able to critically understand the results presented in the literature [5–7]. First, we discovered some errors in Park's calculations, which is why his result for N_c is *not correct*. Second, although the values of N_c obtained by Carlip and by Grumiller et al. are correct, some aspects of the corresponding derivations are *not satisfying*: they are based on introducing an extra constraint by appealing to the Lagrangian formalism, but the effect of this procedure on the overall constraint structure of the theory remains unclear. Our systematic canonical analysis gives a definitive support to the result $N_c = 1$.

After clarifying the constraint structure of TMG_Λ, we extended our analysis to the AdS asymptotic domain. Our study of the subject leads to a remarkable relation between TMG_Λ and 3D gravity with torsion [11]: for a specific choice of parameters which ensures that the torsion vanishes on shell, the conserved charges (energy and angular momentum) and asymptotic symmetries of these two theories *coincide*. This conclusion looks quite natural since it involves, on shell, the Riemannian sector of 3D gravity with torsion. Another interesting aspect of this relation is that it involves two theories with substantially different dynamical contents: 3D gravity with torsion is a topological theory, while TMG_Λ has one propagating degree of freedom.

The paper is organized as follows. In section 2, we give a brief account of the basic dynamical features of TMG_Λ in the Lagrangian formalism. In sections 3 and 4, we apply Dirac's method for constrained dynamical systems [12] to make a complete analysis of the constraint structure of TMG_Λ, which leads to $N_c = 1$. In section 5, we construct a convenient reduced phase space and use it to make a comparison with Carlip's work [6]. The construction of the canonical gauge generator in section 6 confirms the consistency of the previous analysis of constraints. Then, in section 7, we begin the examination of the asymptotic structure of the theory by introducing the AdS asymptotic conditions, which leads to a deep relation between the asymptotic structures of TMG_Λ and 3D gravity with torsion [11]. The full content of this relation is clarified in section 8, devoted to the canonical realization of the asymptotic symmetry: we find the form of the surface term necessary to make the canonical generator well-defined, calculate the conserved charges and identify the central charges of the canonical algebra. Finally, section 9 is devoted to concluding remarks, while appendices contain some technical details.

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Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters $(i, j, k, \ldots; \mu, \nu, \lambda, \ldots)$ run over 0,1,2, the first letters of the Greek alphabet $(\alpha, \beta, \gamma, \ldots)$ run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized as $\varepsilon^{012} = 1$.

2 The Lagrangian dynamics

Topologically massive gravity with a cosmological constant is formulated as a gravitational theory in Riemannian spacetime. Instead of using the standard Riemannian formalism, with an action defined in terms of the metric, we find it more convenient to use the triad field and the spin connection as fundamental dynamical variables. Such an approach can be naturally described in the framework of Poincaré gauge theory [13], where basic gravitational variables are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms), and the corresponding field strengths are the torsion T^i and the curvature R^{ij} (2-forms). After introducing the notation $A^{ij} =: -\varepsilon^{ij}{}_k \omega^k$ and $R^{ij} =: -\varepsilon^{ij}{}_k R^k$, we have:

$$T^i = db^i + \varepsilon^i{}_{jk}\omega^j \wedge b^k\,, \qquad\qquad R^i = d\omega^i + \frac{1}{2}\,\varepsilon^i{}_{jk}\omega^j \wedge \omega^k\,.$$

The antisymmetry of A^{ij} ensures that the underlying geometric structure corresponds to Riemann-Cartan geometry, in which b^i is an orthonormal coframe, $g := \eta_{ij} b^i \otimes b^j$ is the metric of spacetime, ω^i is the Cartan connection, and T^i, R^i are the torsion and the Cartan curvature, respectively. For $T_i = 0$, this geometry reduces to Riemannian. In what follows, we will omit the wedge product sign $\wedge$ for simplicity.

Field equations. The Lagrangian of TMG_Λ is defined by

$$L = 2ab^i R_i - \frac{\Lambda}{3}\,\varepsilon_{ijk} b^i b^j b^k \; + a\mu^{-1} L_{\mathrm{CS}}(\omega) + \lambda^i T_i\,, \tag{2.1}$$

where $a = 1/16\pi G$, $L_{\mathrm{CS}}(\omega) = \omega^i d\omega_i + \frac{1}{3}\varepsilon_{ijk}\omega^i\omega^j\omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection, λ^i (1-form) is the Lagrange multiplier that ensures $T_i = 0$.

The variation of the action $I = \int L$ with respect to b^i, ω^i and λ^i, yields the gravitational field equations:

$$2aR_i - \Lambda\varepsilon_{ijk} b^j b^k + \nabla\lambda_i = 0\,, \tag{2.2a}$$

$$2aT_i + 2a\mu^{-1} R_i + \varepsilon_{imn}\lambda^m b^n = 0\,, \tag{2.2b}$$

$$T_i = 0\,, \tag{2.2c}$$

where $\nabla\lambda_i = d\lambda_i + \varepsilon_{ijk}\omega^j\lambda^k$ is the covariant derivative of λ_i. With $T_i = 0$, the second equation yields a simple solution for λ_m:

$$\lambda_m = 2a\mu^{-1} L_m\,, \qquad\qquad L_m := \left((Ric)_{mn} - \frac{1}{4}\eta_{mn} R\right) b^n\,,$$

where $(Ric)_{mn} = -\varepsilon^{kl}{}_m R_{kln}$, $R = -\varepsilon^{ijk} R_{ijk}$. After that, the first equation takes the form

$$2aR_i - \Lambda\varepsilon_{ijk} b^j b^k + 2a\mu^{-1} C_i = 0\,, \tag{2.3a}$$

where $C_i = \nabla L_i$ is the Cotton 2-form. The expansion in the basis $\hat{\epsilon}_k = \frac{1}{2}\varepsilon_{kmn}b^m b^n$, given by $R_i = G^k{}_i\hat{\epsilon}_k$, $C_i = C^k{}_i\hat{\epsilon}_k$, yields the standard component form of the above equation:

$$aG_{ij} - \Lambda\eta_{ij} + a\mu^{-1}C_{ij} = 0\,, \tag{2.3b}$$

where G_{ij} is the Einstein tensor, and $C_{ij} = \varepsilon_i{}^{mn}\nabla_m L_{nj}$ the Cotton tensor.

For later convenience, we display here two simple consequences of the field equations:

$$\lambda_{mn} - \lambda_{nm} = 0\,, \qquad\qquad \mu\lambda + 3\Lambda = 0\,, \tag{2.4}$$

where $\lambda = \lambda^n{}_n$.

Gauge symmetries. By construction, gauge symmetries of the theory (2.1) are local translations and local Lorentz rotations, parametrized by ξ^μ and $\varepsilon^{ij} =: -\varepsilon^{ij}{}_k\theta^k$. In local coordinates x^μ, we have $b^i = b^i{}_\mu dx^\mu$, $\omega^i = \omega^i{}_\mu dx^\mu$, $\lambda^i = \lambda^i{}_\mu dx^\mu$, and local Poincaré transformations take the form:

$$\begin{aligned}
\delta_P b^i{}_\mu &= -\varepsilon^i{}_{jk}b^j{}_\mu\theta^k - (\partial_\mu\xi^\rho)b^i{}_\rho - \xi^\rho\partial_\rho b^i{}_\mu\,,\\
\delta_P\omega^i{}_\mu &= -\nabla_\mu\theta^i - (\partial_\mu\xi^\rho)\omega^i{}_\rho - \xi^\rho\partial_\rho\omega^i{}_\mu\,,\\
\delta_P\lambda^i{}_\mu &= -\varepsilon^i{}_{jk}\lambda^j{}_\mu\theta^k - (\partial_\mu\xi^\rho)\lambda^i{}_\rho - \xi^\rho\partial_\rho\lambda^i{}_\mu\,.
\end{aligned} \tag{2.5}$$

The BTZ black hole. The BTZ black hole [14], a well-known solution of the standard 3D gravity in the AdS sector (with $\Lambda = -1/\ell^2$), is a trivial solution of TMG_Λ, since the related Cotton tensor identically vanishes.

In the Schwartzschield-like coordinates $x^\mu = (t, r, \varphi)$, the BTZ black hole solution is defined in terms of the lapse and shift functions, respectively:

$$N^2 = \left(-8Gm + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2}\right)\,, \qquad\qquad N_\varphi = \frac{4GJ}{r^2}\,.$$

The triad field has the simple diagonal form

$$b^0 = N dt\,, \qquad\qquad b^1 = N^{-1}dr\,, \qquad\qquad b^2 = r\left(d\varphi + N_\varphi dt\right)\,, \tag{2.6a}$$

the connection reads

$$\omega^0 = -N d\varphi\,, \qquad\qquad \omega^1 = N^{-1}N_\varphi dr\,, \qquad\qquad \omega^2 = -\frac{r}{\ell^2}dt - N_\varphi r d\varphi\,, \tag{2.6b}$$

and the Lagrange multiplier is expressed in terms of the triad field as

$$\lambda^i = \frac{a}{\mu\ell^2}b^i\,. \tag{2.6c}$$

Maximally symmetric solution of TMG_Λ, the AdS solution with isometry group $\mathrm{SO}(2,2)$, is formally obtained from (2.6) by the replacements $8mG = -1$, $J = 0$.

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3 Hamiltonian and constraints

In order to get a deeper insight into dynamical structure of TMG_Λ, we focus our attention on its canonical content [12]. In local coordinates x^μ, the component form of the Lagrangian density reads:

$$\mathcal{L} = \varepsilon^{\mu\nu\rho}\left[ab^i{}_\mu R_{i\nu\rho} - \frac{\Lambda}{3}\varepsilon_{ijk}b^i{}_\mu b^j{}_\nu b^k{}_\rho + a\mu^{-1}\left(\omega^i{}_\mu\partial_\nu\omega_{i\rho} + \frac{1}{3}\varepsilon_{ijk}\omega^i{}_\mu\omega^j{}_\nu\omega^k{}_\rho\right) + \frac{1}{2}\lambda^i{}_\mu T_{i\nu\rho}\right].$$

1. Introducing the canonical momenta $(\pi_i{}^\mu, \Pi_i{}^\mu, p_i{}^\mu)$ corresponding to the Lagrangian variables $(b^i{}_\mu, \omega^i{}_\mu, \lambda^i{}_\mu)$, we find the primary constraints:

$$\begin{aligned} \phi_i{}^0 &:= \pi_i{}^0 \approx 0\,, & \phi_i{}^\alpha &:= \pi_i{}^\alpha - \varepsilon^{0\alpha\beta}\lambda_{i\beta} \approx 0\,,\\ \Phi_i{}^0 &:= \Pi_i{}^0 \approx 0\,, & \Phi_i{}^\alpha &:= \Pi_i{}^\alpha - a\varepsilon^{0\alpha\beta}(2b_{i\beta} + \mu^{-1}\omega_{i\beta}) \approx 0\,.\\ p_i{}^\mu &\approx 0\,. \end{aligned} \tag{3.1}$$

The canonical Hamiltonian has the form:

$$\begin{aligned} \mathcal{H}_c &= b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + \lambda^i{}_0\mathcal{T}^i + \partial_\alpha D^\alpha\,,\\ \mathcal{H}_i &= -\varepsilon^{0\alpha\beta}\left(aR_{i\alpha\beta} - \Lambda\varepsilon_{ijk}b^j{}_\alpha b^k{}_\beta + \nabla_\alpha\lambda_{i\beta}\right),\\ \mathcal{K}_i &= -\varepsilon^{0\alpha\beta}\left(aT_{i\alpha\beta} + a\mu^{-1}R_{i\alpha\beta} + \varepsilon_{ijk}b^j{}_\alpha\lambda^k{}_\beta\right),\\ \mathcal{T}_i &= -\frac{1}{2}\varepsilon^{0\alpha\beta}T_{i\alpha\beta}\,,\\ D^\alpha &= \varepsilon^{0\alpha\beta}\left[a\omega^i{}_0\left(2b_{i\beta} + \mu^{-1}\omega_{i\beta}\right) + b^i{}_0\lambda_{i\beta}\right]. \end{aligned}$$

The basic Poisson brackets (PBs) are displayed in appendix A.

2. Going over to the total Hamiltonian,

$$\mathcal{H}_T = b^i{}_0\mathcal{H}_i + \omega^i{}_0\mathcal{K}_i + \lambda^i{}_0\mathcal{T}_i + u^i{}_\mu\phi_i{}^\mu + v^i{}_\mu\Phi_i{}^\mu + w^i{}_\mu p_i{}^\mu + \partial_\alpha D^\alpha\,, \tag{3.2}$$

we find that the consistency conditions of the primary constraints $\pi_i{}^0$, $\Pi_i{}^0$ and $p_i{}^0$ yield the secondary constraints:

$$\mathcal{H}_i \approx 0, \qquad \mathcal{K}_i \approx 0\,, \qquad \mathcal{T}_i \approx 0\,. \tag{3.3a}$$

The consistency of the remaining primary constraints $\phi_i{}^\alpha$, $\Phi_i{}^\alpha$ and $p_i{}^\alpha$ leads to the determination of the multipliers $u^i{}_\beta$, $v^i{}_\beta$ and $w^i{}_\beta$. Denoting the determined multipliers by a bar, we have:

$$\begin{aligned} 2a(\bar{v}_{i\beta} - \nabla_\beta\omega_{i0}) + \bar{w}_{i\beta} + \varepsilon_{ijk}\omega^j{}_0\lambda^k{}_\beta - \nabla_\beta\lambda_{i0} - 2\Lambda\varepsilon_{ijk}b^j{}_0b^k{}_\beta &= 0\,,\\ 2a\mu^{-1}(\bar{v}_{i\beta} - \nabla_\beta\omega_{i0}) + \varepsilon_{ijk}(b^j{}_0\lambda^k{}_\beta - b^j{}_\beta\lambda^k{}_0) &= 0\,,\\ \bar{u}_{i\beta} + \varepsilon_{ijk}\omega^j{}_0b^k{}_\beta - \nabla_\beta b_{i0} &= 0\,. \end{aligned} \tag{3.3b}$$

Using the Hamiltonian equations of motion $\dot{b}^i{}_\beta = \bar{u}^i{}_\beta$, $\dot{\omega}^i{}_\beta = \bar{v}^i{}_\beta$ and $\dot{\lambda}^i{}_\beta = \bar{w}^i{}_\beta$, these relations reduce to the $(0,\beta)$ components of the Lagrangian field equations (2.2).

The substitution of the determined multipliers into (3.2) yields the modified form of the total Hamiltonian:

$$\begin{aligned}\mathcal{H}_T &= \hat{\mathcal{H}}_T + \partial_\alpha \bar{D}^\alpha\,,\\ \hat{\mathcal{H}}_T &= b^i{}_0\bar{\mathcal{H}}_i + \omega^i{}_0\bar{\mathcal{K}}_i + \lambda^i{}_0\bar{\mathcal{T}}_i + u^i{}_0\pi_i{}^0 + v^i{}_0\Pi_i{}^0 + w^i{}_0 p_i{}^0\,,\end{aligned}$$

where

$$\begin{aligned}\bar{\mathcal{H}}_i &= \mathcal{H}_i - \nabla_\beta\phi_i{}^\beta - \frac{\mu}{2a}\varepsilon_{ijk}\lambda^j{}_\beta\Phi^{k\beta} + \varepsilon_{ijk}\left(2\Lambda b^j{}_\beta + \mu\lambda^j{}_\beta\right)p^{k\beta}\,,\\ \bar{\mathcal{K}}_i &= \mathcal{K}_i - \varepsilon_{ijk}b^j{}_\beta\phi^{k\beta} - \nabla_\beta\Phi_i{}^\beta - \varepsilon_{ijk}\lambda^j{}_\beta p^{k\beta}\,,\\ \bar{\mathcal{T}}_i &= \mathcal{T}_i - \frac{\mu}{2a}\varepsilon_{ijk}b^j{}_\beta\Phi^{k\beta} - \nabla_\beta p_i{}^\beta + \mu\varepsilon_{ijk}b^j{}_\beta p^{k\beta}\,,\\ \bar{D}^\alpha &= D^\alpha + b^i{}_0\phi_i{}^\alpha + \omega^i{}_0\Phi_i{}^\alpha + \lambda^i{}_0 p_i{}^\alpha\,.\end{aligned}$$

3. The consistency conditions of the secondary constraints read:

$$\begin{aligned}\{\bar{\mathcal{H}}_i, H_T\} &\approx -\frac{\mu}{2a}\varepsilon^{0\alpha\beta}\Big[b_{i0}\lambda_{\alpha\beta} - \lambda_{i\alpha}(\lambda_{0\beta} - \lambda_{\beta 0})\Big] =: X_i\,,\\ \{\bar{\mathcal{T}}_i, H_T\} &\approx \frac{\mu}{2a}\varepsilon^{0\alpha\beta}\Big[b_{i0}\lambda_{\alpha\beta} - b_{i\alpha}(\lambda_{0\beta} - \lambda_{\beta 0})\Big] =: Y_i\,,\\ \{\bar{\mathcal{K}}_i, H_T\} &\approx 0\,,\end{aligned} \tag{3.4}$$

where $\lambda_{\mu\nu} = b^k{}_\mu\lambda_{k\nu}$. This result contains an important difference with respect to the one obtained by Park, eq. (14) in [5], which consists in the presence of the $\lambda_{\alpha\beta}$ terms. To count the number of independent tertiary constraints, one notes that $Y_i \approx 0$ is equivalent to

$$\theta_{0\beta} := \lambda_{0\beta} - \lambda_{\beta 0} \approx 0\,, \tag{3.5a}$$

$$\theta_{\alpha\beta} := \lambda_{\alpha\beta} - \lambda_{\beta\alpha} \approx 0\,, \tag{3.5b}$$

which, in turn, ensures $X_i \approx 0$. Thus, we have only *three independent* tertiary constraints, $\theta_{0\beta}$ and $\theta_{\alpha\beta}$, which are the canonical equivalents of the Lagrangian relations $(2.4)_1$.

4. The consistency of $\theta_{\alpha\beta}$ yields

$$\{\theta_{\alpha\beta}, H_T\} = \bar{u}^k{}_\alpha\lambda_{k\beta} + b^k{}_\alpha\bar{w}_{k\beta} - (\alpha\leftrightarrow\beta) \approx -2b\varepsilon_{0\alpha\beta}\Big(3\Lambda + \mu\lambda\Big) \approx 0\,.$$

Thus, we have a new, quartic constraint:

$$\Psi = 3\Lambda + \mu\lambda \approx 0\,. \tag{3.6}$$

The quartic constraint is a canonical equivalent of the Lagrangian relation $(2.4)_2$.

To interpret the consistency condition for $\theta_{0\beta}$, we introduce the notation

$$\pi_i{}^{0\prime} := \pi_i{}^0 + \lambda_i{}^k p_k{}^0\,, \qquad\qquad w^i{}_0{}' := w^i{}_0 - u^k{}_0\lambda_k{}^i\,. \tag{3.7}$$

The $(\pi_i{}^0, p_i{}^0)$ piece of the Hamiltonian can be written in the form

$$u^i{}_0\pi_i{}^0 + w^i{}_0 p_i{}^0 = u^i{}_0\pi_i{}^{0\prime} + w^i{}_0{}' p_i{}^0\,.$$

The consistency of $\theta_{0\beta}$ imposes a condition on the two components $w'_{\beta 0} = w_{m0}{}'b^m{}_\beta$ of $w_i{}^{0\prime}$:

$$\begin{aligned}\{\theta_{0\beta}, H_T\} &= (b^i{}_0\bar{w}_{i\beta} - \lambda^i{}_0\bar{u}_{i\beta}) - (w^m{}_0 - u^k{}_0\lambda_k{}^m)b_{m\beta} \approx 0\,,\\ \bar{w}'_{\beta 0} &= b^i{}_0\bar{w}_{i\beta} - \lambda^i{}_0\bar{u}_{i\beta}\,.\end{aligned} \tag{3.8}$$

5. Finally, the consistency requirement on Ψ determines $w'_{00} = w'_{m0}b^m{}_0$:

$$\begin{aligned}\{\Psi, H_T\} &= g^{00}\bar{w}'_{00} + g^{\beta 0}\bar{w}'_{\beta 0} + h^{i\beta}(\bar{w}_{i\beta} - \lambda_i{}^k\bar{u}_{k\beta}) \approx 0\,,\\ g^{00}\bar{w}'_{00} &= (\lambda^{i\beta} + \lambda^i{}_0 g^{0\beta})\bar{u}_{i\beta} - (h^{i\beta} + b^i{}_0 g^{0\beta})\bar{w}_{i\beta}\,.\end{aligned} \tag{3.9}$$

This completes the consistency procedure.

The final form of the total Hamiltonian can be written as

$$\begin{aligned}\hat{\mathcal{H}}_T &= \bar{\mathcal{H}}_T + u^i{}_0\pi_i{}^{0\prime} + v^i{}_0\Pi_i{}^0\,,\\ \bar{\mathcal{H}}_T &:= b^i{}_0\bar{\mathcal{H}}_i + \omega^i{}_0\bar{\mathcal{K}}_i + \lambda^i{}_0\bar{\mathcal{T}}_i + \bar{w}'_{\beta 0}p^{\beta 0} + \bar{w}'_{00}p^{00}\,.\end{aligned} \tag{3.10}$$

4 Classification of constraints

Among the primary constraints, those that appear in $\mathcal{H}_T$ with arbitrary multipliers are first class (FC):

$$\pi_i{}^{0\prime}, \Pi_i{}^0 = \text{FC}\,, \tag{4.1a}$$

while the remaining ones are second class.

Going to the secondary constraints, we use the following simple theorem:

- If ϕ is a FC constraint, then $\{\phi, H_T\}$ is also a FC constraint.

The proof relies on using the Jacoby identity. The theorem implies that the secondary constraints $\hat{\mathcal{H}}_i := -\{\pi_i{}^{0\prime}, H_T\}$ and $\hat{\mathcal{K}}_i := -\{\Pi_i{}^0, H_T\}$ are FC. After a lengthy but straightforward calculation, we obtain:

$$\begin{aligned}\hat{\mathcal{H}}_i &= \bar{\mathcal{H}}'_i + h_i{}^\rho(\nabla_\rho\lambda_{jk})b^k{}_0p^{j0}\,,\\ \hat{\mathcal{K}}_i &= \bar{\mathcal{K}}_i - \varepsilon_{ijk}(\lambda^j{}_0p^{k0} - b^j{}_0\lambda^k{}_np^{n0})\,,\end{aligned} \tag{4.1b}$$

where $\bar{\mathcal{H}}' := \bar{\mathcal{H}}_i + \lambda_i{}^k\bar{\mathcal{T}}_k$. In deriving the above form of $\hat{\mathcal{H}}_i$, we used the weak equality

$$2\Lambda\varepsilon_{inm} + \mu\left(\varepsilon_{ink}\lambda^k{}_m - \varepsilon_{imk}\lambda^k{}_n\right) \approx h_n{}^\mu h_m{}^\nu(\nabla_\mu\lambda_{i\nu} - \nabla_\nu\lambda_{i\mu})\,,$$

where time derivatives are expressed in terms of the determined multipliers.

The PB algebra between the FC constraints $(\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_j)$ is calculated in appendix A. The total Hamiltonian can be expressed in terms of the FC constraints as follows:

$$\hat{\mathcal{H}}_T = b^i{}_0\hat{\mathcal{H}}_i + \omega^i{}_0\hat{\mathcal{K}}_i + u^i{}_0\pi_i{}^{0\prime} + v^i{}_0\Pi_i{}^0 - \theta_{0\beta}h^{n\beta}\hat{\mathcal{T}}_n\,, \tag{4.2}$$

where the last term is an ignorable square of constraints, with

$$\hat{\mathcal{T}}_n := -\{\pi_n{}^0, H_T\} = \bar{\mathcal{T}}_n - b_{n0}\partial_\beta p^{\beta 0} - (\nabla_\beta b_{n0})p^{\beta 0} - \mu\varepsilon_{njk}b^j{}_0b^k{}_\beta p^{\beta 0}\,.$$

The complete classification of constraints is summarized in table 1.

	First class	Second class
Primary	${\pi_i}^{0\prime}, {\Pi_i}^0$	${\phi_i}^\alpha, {\Phi_i}^\alpha, {p_i}^\alpha, {p_i}^0$
Secondary	$\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_i$	$\bar{\mathcal{T}}_i$
Tertiary		$\theta_{0\beta}, \theta_{\alpha\beta}$
Quartic		Ψ

Table 1. Classification of contraints.

The content of table 1 related to the second class constraints needs additional explanation. We begin by noting that the primary constraints $({\phi_i}^\alpha, {\Phi_i}^\alpha, {p_i}^\alpha, {p_i}^0)$ are of the second class, as the related multipliers in H_T are determined. The second class nature of the remaining constraints $(\bar{\mathcal{T}}_i, \theta_{0\beta}, \theta_{\alpha\beta}, \Psi)$ can be verified by analyzing their PB algebra. There is, however, a much simpler argument based on the counting of dynamical degrees of freedom, as explained bellow.

When the classification of constraints is complete, the number of dynamical degrees of freedom in the phase space is given by the formula:

$$N^* = 2N - 2N_1 - N_2\,,$$

where N is the number of Lagrangian dynamical variables, N_1 is the number of FC, and N_2 the number of second class constraints. According to our results, we have $N = 27$, $N_1 = 12$ and $N_2 = 28$, the dimension of the phase space is $N^* = 2$, and the theory exhibits one local Lagrangian degree of freedom, the topologically massive graviton [2, 8].

The argument that supports the classification displayed in table 1 goes as follows. If at least two constraints in the set $(\bar{\mathcal{T}}_i, \theta_{0\beta}, \theta_{\alpha\beta}, \Psi)$ were FC, then N^* would be negative. This is, however, *not possible*, hence, all the constraints $(\bar{\mathcal{T}}_i, \theta_{0\beta}, \theta_{\alpha\beta}, \Psi)$ are of the second class. A more technical argument on this point is given in the next section.

5 The reduced phase space

The canonical analysis of TMG_Λ developed so far is based on using the full phase space with coordinates $({b^i}_\mu, {\omega^i}_\mu, {\lambda^i}_\mu; {\pi_i}^\mu, {\Pi_i}^\mu, {p_i}^\mu)$. Now, we wish to examine what happens when we go to the reduced phase space formalism, in which the PBs are replaced by the Dirac brackets (DB) [12].

We begin by noting that we have two sets of FC constraints, ${\pi_i}^{0\prime}$ and ${\Pi_i}^0$, hence we are free to impose two sets of gauge conditions. A simple and natural choice is to fix the form of the corresponding unphysical variables, ${b^i}_0$ and ${\omega^i}_0$. This can be done, for instance, by demanding their forms to coincide with the black hole solution. After that, we can construct the corresponding DBs and eliminate the variables $({b^i}_0, {\pi_i}^{0\prime})$ and $({\omega^i}_0, {\Pi_i}^0)$ from the theory; the DBs of the remaining variables remain unchanged. Note that similar arguments cannot be applied to the pair $({p_i}^0, {\lambda^i}_0)$, since ${p_i}^0$ is not a FC constraint.

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	First class	Second class
Secondary	$\mathcal{H}'_i, \mathcal{K}_i$	$\mathcal{T}_i$
Tertiary		$\theta_{\alpha\beta}$

Table 2. Classification of constraints in R_2.

Next, we use the second class constraints $X_A := (\phi_i{}^\alpha, \Phi_i{}^\alpha, p_i{}^\alpha)$ to eliminate the remaining momenta $(\pi_i{}^\alpha, \Pi_i{}^\alpha, p_i{}^\alpha)$. After that, the structure of the reduced phase space R_1 with canonical coordinates $(b^i{}_\alpha, \omega^i{}_\alpha, \lambda^i{}_\alpha; \lambda^i{}_0, p_i{}^0)$ is determined by the DBs

$$\begin{aligned}
&\{b^i{}_\alpha, b^j{}_\beta\}^*_1 = 0\,, && \{b^i{}_\alpha, \omega^j{}_\beta\}^*_1 = 0\,, && \{b^i{}_\alpha, \lambda^j{}_\beta\}^*_1 = \varepsilon_{0\alpha\beta}\eta^{ij}\delta\,,\\
&\{\omega^i{}_\alpha, \omega^j{}_\beta\}^*_1 = \frac{\mu}{2a}\varepsilon_{0\alpha\beta}\eta^{ij}\delta\,, && \{\omega^i{}_\alpha, \lambda^j{}_\beta\}^*_1 = -\mu\varepsilon_{0\alpha\beta}\eta^{ij}\delta\,,\\
&\{\lambda^i{}_\alpha, \lambda^j{}_\beta\}^*_1 = 2a\mu\varepsilon_{0\alpha\beta}\eta^{ij}\delta\,,
\end{aligned} \tag{5.1}$$

plus those involving $\lambda^i{}_0$ and $p_i{}^0$ (appendix B).

Finally, we introduce the reduced phase space R_2, defined by the 6 second class constraints $Y_A := (\theta_{0\beta}, \Psi, p^{\alpha 0}, p_0{}^0)$. The constraints Y_A can be used to eliminate $\lambda^i{}_0$ and $p_i{}^0$ from R_1, whereupon the reduced phase space R_2 is described by the canonical coordinates $(b^i{}_\alpha, \omega^i{}_\alpha, \lambda^i{}_\alpha)$. Using the iterative property of DBs, the influence of Y_A on the form of DBs is described by the matrix Δ_2, with $(\Delta_2)_{AB} = \{Y_A, Y_B\}^*_1$ (appendix B). Explicit calculation shows that the form of the new DBs is defined by the following simple rule:

- The new DBs in R_2 are the same as those in eq. (5.1).

The classification of constraints in R_2 is displayed in table 2.

The number of the phase space variables is $3 \times 6 = 18$, there are 6 first class and 4 second class constraints, and the number of physical degrees of freedom is the same as before, $N^* = 18 - 2 \times 6 - 4 = 2$, as it should.

Treating $(b^i{}_0, \omega^i{}_0, \lambda^i{}_0)$ as Lagrange multipliers, Carlip worked from the very beginning in the reduced phase with canonical coordinates $(b^i{}_\alpha, \omega^i{}_\alpha, \lambda^i{}_\alpha)$ [6]. To compare his construction with our R_2, we replace the variables ω^i and λ^i by $A^i = \omega^i + \mu b^i$ and $\beta^i = \lambda^i - a\mu b^i$, respectively. The resulting non-trivial DBs are:

$$\{A^i{}_\alpha, A^j{}_\beta\}^*_2 = \frac{\mu}{2a}\varepsilon_{0\alpha\beta}\eta^{ij}\delta\,, \qquad \{b^i{}_\alpha, \beta^j{}_\beta\}^*_2 = \varepsilon_{0\alpha\beta}\eta^{ij}\delta\,, \tag{5.2}$$

in complete agreement with eq. (3.2) in [6] (in units $a = 1$). Hence, R_2 coincides with Carlip's construction of the phase space.

At this stage, one can check the second class nature of $Z_A = (\mathcal{T}_i, \theta_{\alpha\beta})$ directly from the form of their DBs:

$$\begin{aligned}
\{\mathcal{T}_i, \mathcal{T}_j\}^*_2 &= \frac{\mu}{2a}\varepsilon^{0\alpha\beta} b_{i\alpha} b_{j\beta}\,,\\
\{\mathcal{T}_i, \theta_{\alpha\beta}\}^*_2 &= \nabla_\beta(b_{i\alpha}\delta) - \nabla_\alpha(b_{i\beta}\delta) + 2\mu\varepsilon_{imn} b^m{}_\alpha b^n{}_\beta\delta\,,\\
\{\theta_{\alpha\beta}, \theta_{\gamma\delta}\}^*_2 &= 0\,.
\end{aligned} \tag{5.3}$$

Indeed, as shown in [6], the matrix $(\Delta_3)_{AB} = \{Z_A, Z_B\}^*_2$ is invertible.

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6 Gauge generator

After completing the Hamiltonian analysis, we now wish to construct the canonical gauge generator [15]. Starting from the primary FC constraints $\pi_i{}^{0\prime}$ and $\Pi_i{}^0$, one finds:

$$\begin{aligned} G[\tau] &= \dot{\tau}^i \pi_i{}^{0\prime} + \tau^i \left[\hat{\mathcal{H}}^i - \varepsilon_{ijk}\omega^j{}_0 \pi^{k0\prime} + \frac{\mu}{2a}\left(\varepsilon_{imn}\lambda_j{}^n - \varepsilon_{jmn}\lambda_i{}^n\right) b^j{}_0 \Pi^{m0}\right] , \\ G[\sigma] &= \dot{\sigma}^i \Pi_i{}^0 + \sigma^i \left(\hat{\mathcal{K}}^i - \varepsilon_{ijk}\omega^j{}_0 \Pi^k{}_0 - \varepsilon_{ijk} b^j{}_0 \pi^{k0\prime}\right) . \end{aligned} \tag{6.1}$$

The complete gauge generator has the form $G = G[\tau] + G[\sigma]$, its action on the fields is defined by the PB operation $\delta_0\phi = \{\phi, G\}$, but the resulting gauge transformations do not have the Poincaré form (2.5). The standard Poincaré content of the gauge transformations is obtained by introducing the new parameters [11]

$$\tau^i = -\xi^\rho b^i{}_\rho , \qquad\qquad \sigma^i = -\theta^i - \xi^\rho \omega^i{}_\rho .$$

Expressed in terms of these parameters (and after neglecting some trivial terms, quadratic in the constraints), the gauge generator takes the form:

$$\begin{aligned} G &= -G_1 - G_2 , \\ G_1 &= \dot{\xi}^\rho \left(b^i{}_\rho \pi_i{}^0 + \lambda^i{}_\rho p_i{}^0 + \omega^i{}_\rho \Pi_i{}^0\right) \\ &\quad + \xi^\rho \left[b^i{}_\rho \bar{\mathcal{H}}_i + \lambda^i{}_\rho \bar{\mathcal{T}}_i + \omega^i{}_\rho \bar{\mathcal{K}}_i + (\partial_\rho b_0^i)\pi_i{}^0 + (\partial_\rho \lambda^i{}_0) p_i{}^0 + (\partial_\rho \omega^i{}_0)\Pi^i{}_0\right] , \\ G_2 &= \dot{\theta}^i \Pi_i{}^0 + \theta^i \left[\bar{\mathcal{K}}_i - \varepsilon_{ijk}\left(b^j{}_0 \pi^{k0} + \lambda^j{}_0 p^{k0} + \omega^j{}_0 \Pi^{k0}\right)\right] . \end{aligned} \tag{6.2}$$

Looking at the related gauge transformations, we find a complete agreement with the Poincaré gauge transformations (2.5) *on shell*.

7 Asymptotic conditions

Asymptotic conditions imposed on dynamical variables determine the form of asymptotic symmetries, and consequently, they are closely related to the gravitational conservation laws. In this section, we focus our attention to the AdS sector of the theory, characterized by the negative value of the cosmological constant:

$$\frac{\Lambda}{a} =: -\frac{1}{\ell^2} .$$

AdS asymptotics. The AdS asymptotic conditions are introduced by demanding that (a) the asymptotic configurations include the black hole solution (2.6), and (b) they are invariant under the action of the AdS group SO(2, 2). Following the procedure defined in 3D gravity with torsion [11], we find the asymptotic form for the triad field:

$$b^i{}_\mu = \begin{pmatrix} \frac{r}{\ell} + \mathcal{O}_1 & O_4 & O_1 \\ \mathcal{O}_2 & \frac{\ell}{r} + \mathcal{O}_3 & O_2 \\ \mathcal{O}_1 & O_4 & r + \mathcal{O}_1 \end{pmatrix} , \tag{7.1a}$$

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and for the connection:

$$\omega^i{}_\mu = \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & -\frac{r}{\ell}+\mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_2 \\ -\frac{r}{\ell^2}+\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1 \end{pmatrix} . \tag{7.1b}$$

In TMG_Λ, we have one more Lagrangian variable, the Lagrange multiplier λ^i. Since λ^i for the black hole solution satisfies (2.6c), we define its asymptotic behavior by the relation:

$$\lambda^i{}_\mu = \frac{a}{\mu\ell^2} b^i{}_\mu + \hat{\mathcal{O}} \,, \tag{7.1c}$$

where $\hat{\mathcal{O}}$ denotes terms with arbitrarily fast asymptotic decrease.

At this stage, by comparing (7.1a) and (7.1b) with the asymptotic conditions in 3D gravity with torsion, see section 4 in [11], we are led to an important observation:

(A1) The asymptotic form of $b^i{}_\mu$ and $\omega^i{}_\mu$ in TMG_Λ is the same as in 3D gravity with torsion in the limit when the torsion vanishes on shell.

Looking at the field equations of 3D gravity with torsion displayed in appendix C, one finds that the condition of vanishing torsion takes the form $p = 0$, where p is a combination of the coupling constants. The origin of this property may be traced back to the form of the BTZ black hole (2.6). As we shall see in the next section, (A1) lies at the root of a remarkable correspondence between the asymptotic structures of TMG_Λ and 3D gravity with torsion.

Asymptotic parameters. Having chosen the asymptotic conditions in the form (7.1), we now wish to find the subset of gauge transformations that respect these conditions. As a first consequence of (A1), we conclude that the parameters of the restricted gauge transformations have the same form as in 3D gravity with torsion [11]:

$$\begin{aligned} \xi^0 &= \ell\left[T + \frac{1}{2}\left(\frac{\partial^2 T}{\partial t^2}\right)\frac{\ell^4}{r^2}\right] + \mathcal{O}_4 \,, \qquad & \xi^1 &= -\ell\left(\frac{\partial T}{\partial t}\right) r + \mathcal{O}_1 \,, \\ \xi^2 &= S - \frac{1}{2}\left(\frac{\partial^2 S}{\partial \varphi^2}\right)\frac{\ell^2}{r^2} + \mathcal{O}_4 \,, \end{aligned} \tag{7.2}$$

and similarly for θ^i. Here, the functions $T(t,\varphi)$ and $S(t,\varphi)$ are determined by the conditions

$$T^- = T^-(x^-) \,, \qquad T^+ = T^+(x^+) \,,$$

where $T^\mp = T \mp S$ and $x^\mp = x^0/\ell \mp x^2$. After expressing $T^\mp$ in terms of the Fourier modes and introducing the notation $\delta_P(T^\mp = e^{inx^\mp}) =: \ell^\mp_n$, the asymptotic commutator algebra takes the familiar form of two independent Virasoro algebras without central charges:

$$i[\ell^-_n, \ell^-_m] = (n-m)\ell^-_{n+m} \,, \qquad i[\ell^+_n, \ell^+_m] = (n-m)\ell^+_{n+m} \,.$$

The asymptotic symmetry of spacetime, defined by the parameters $T^\mp$, coincides with the conformal symmetry.

Asymptotics of the phase space. In order to extend the asymptotic conditions (7.1) to the canonical level, one should determine an appropriate asymptotic behavior of the momentum variables. This step is based on the following general principle: the expressions than vanish on shell should have an arbitrary fast asymptotic decrease, as no solutions of the field equations are thereby lost. By applying this principle to the primary constraints (3.1), one finds the asymptotic behavior of all the momentum variables.

8 Canonical realization of the asymptotic symmetry

In this section, we study the influence of the adopted asymptotic conditions on the canonical structure of TMG_Λ: we construct the improved gauge generators, examine their canonical algebra and prove the conservation laws. As a consequence of (A1), all these characteristics are naturally related the the corresponding results in 3D gravity with torsion.

8.1 Surface terms

The canonical generator acts on dynamical variables via the PB operation, hence, it should have well-defined functional derivatives. In order to ensure this property, we have to improve the form of G by adding a suitable surface term Γ, such that $\tilde{G} = G + \Gamma$ is a well-defined canonical generator. In this process, the asymptotic conditions play a crucial role [11, 16].

Following the same calculational technique as in [11], we find that the improved canonical generator takes the form

$$\tilde{G} = G + \Gamma\,,$$
$$\Gamma := -\oint df_\alpha \left(\xi^0 \mathcal{E}^\alpha + \xi^2 \mathcal{M}^\alpha\right) = -\int_0^{2\pi} d\varphi \left(\ell T \mathcal{E}^1 + S\mathcal{M}^1\right)\,, \tag{8.1a}$$

where

$$\mathcal{E}^\alpha = 2\varepsilon^{0\alpha\beta}\left(a\omega^0{}_\beta + \frac{a}{2\mu\ell^2}b^0{}_\beta + \frac{1}{2}\lambda^0{}_\beta + \frac{a}{\ell}b^2{}_\beta + \frac{a}{\mu\ell}\omega^2{}_\beta\right)b^0{}_0\,,$$
$$\mathcal{M}^\alpha = -2\varepsilon^{0\alpha\beta}\left(a\omega^2{}_\beta + \frac{a}{2\mu\ell^2}b^2{}_\beta + \frac{1}{2}\lambda^2{}_\beta + \frac{a}{\ell}b^0{}_\beta + \frac{a}{\mu\ell}\omega^0{}_\beta\right)b^2{}_2\,. \tag{8.1b}$$

Now, we can use the asymptotic relation (7.1c) for $\lambda^i{}_\mu$ and compare the value of the surface term Γ with the corresponding expression for 3D gravity with torsion, displayed in appendix C, with the following conclusion:

(A2) The value of the surface integral Γ in the AdS sector of TMG_Λ coincides with the corresponding value in 3D gravity with torsion, in the limit of vanishing torsion.

This conclusion is a natural consequence of (A1).

8.2 Conserved charges

The values of the surface terms, calculated for $\xi^0 = 1$ and $\xi^2 = 1$, define the energy and angular momentum of the system, respectively:

$$E = \int_0^{2\pi} d\varphi\, \mathcal{E}^1\,, \qquad\qquad M = \int_0^{2\pi} d\varphi\, \mathcal{M}^1\,. \tag{8.2}$$

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In particular, the energy and angular momentum for the BTZ black hole (2.6) are:

$$E = m - \frac{J}{\mu\ell^2}\,, \qquad\qquad M = J - \frac{m}{\mu}\,. \tag{8.3}$$

In agreement with (A2), these BTZ charges are seen to coincide with the corresponding expressions in 3D gravity with torsion, in the limit of vanishing torsion, see appendix C.

8.3 Canonical algebra

Using the notation $\tilde{G}_{(i)} := \tilde{G}[T_i^+, T_i^-]$, the main theorem of [17] states that the canonical algebra of the improved generators has the general form:

$$\left\{\tilde{G}_{(2)}, \tilde{G}_{(1)}\right\} = \tilde{G}_{(3)} + C_{(3)}\,, \tag{8.4a}$$

where $C_{(3)}$ is the central term. To calculate $C_{(3)}$, we note that

$$\{\tilde{G}_{(2)}, \tilde{G}_{(1)}\} \approx \delta_{(1)}\Gamma_{(2)} \approx \Gamma_{(3)} + C_{(3)}\,.$$

The calculation of $\delta_{(1)}\Gamma_{(2)}$ is based on the asymptotic transformation laws of the energy/angular momentum densities $\mathcal{E}_\mp = (\ell\mathcal{E}^1 \mp \mathcal{M}^1)/2$:

$$\delta\mathcal{E}_\mp = -T^\mp\partial_\mp\mathcal{E}_\mp - 2(\partial_\mp T^\mp)\mathcal{E}_\mp + a\ell\left(1 \pm \frac{1}{\ell\mu}\right)\partial_\mp^3 T^\mp\,,$$

and it leads to

$$\begin{aligned} C_{(3)} \;&=\; C_-[T^-] + C_+[T^+]\,, \\ C_\mp[T^\mp] \;&:=\; -a\ell\left(1 \pm \frac{1}{\ell\mu}\right)\int_0^{2\pi} d\varphi(\partial_\mp^3 T_1^\mp)T_2^\mp\,. \end{aligned} \tag{8.4b}$$

Introducing the Fourier modes for the improved generator, $L_n^\mp = -\tilde{G}[T^\mp = e^{inx^\mp}]$, the canonical algebra (8.4) takes the form of two independent Virasoro algebras with different central charges:

$$c^\mp = 24\pi a\ell\left(1 \pm \frac{1}{\ell\mu}\right) = \frac{3\ell}{2G}\left(1 \pm \frac{1}{\ell\mu}\right)\,. \tag{8.5}$$

A direct comparison with appendix C implies that the central charges of TMG_Λ have the same values as in the $p = 0$ limit of 3D gravity with torsion, which is, again, a consequence of the general correspondence (A2).

Once we have the central charges, we can use Cardy's formula to obtain the black hole entropy [4, 18]:

$$S = \frac{2\pi r_+}{4G} - \frac{2\pi r_-}{4G\mu\ell}\,, \tag{8.6}$$

where r_+ and r_- (the radii of the outer and inner black hole horizon, respectively) are related to the black hole parameters m and J by $r_+^2 + r_-^2 = 8Gm\ell^2$, $r_+r_- = 4GJ\ell$. The form of the entropy is in agreement with the first law of black hole thermodynamics.

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9 Concluding remarks

In this paper, we studied TMG_Λ as a constrained dynamical system [11]. Our approach is based on using the triad field b^i and the spin connection ω^i as independent dynamical variables, while the Lagrange multiplier λ^i is introduced to ensure the vanishing of torsion. Our goal was twofold: first, to obtain and classify the constraints and deduce the dimension of the physical phase space N^*, and second, to examine the asymptotic structure of TMG_Λ and compare it with the corresponding features of 3D gravity with torsion.

(1) With regard to the first goal, we found $N^* = 2$, which means that the number of Lagrangian degrees of freedom is $N_c = 1$.

Since Park [5] used the same formalism, we can easily compare his results with ours. Park's consistency conditions for $\bar{\mathcal{H}}_i$ and $\bar{\mathcal{T}}_i$ in eq. (14) of [5] are not correctly calculated, as one can see by comparing with our eqs. (3.4) and (3.5). As a consequence, Park missed the tertiary constraint $\theta_{\alpha\beta}$. Without $\theta_{\alpha\beta}$, he was not able to find the quartic constraint Ψ, given in our eq. (3.6). Moreover, one can directly conclude that Park's classification of constraints is not correct. Indeed, if Ψ_a^0 and $\bar{\mathcal{K}}_a$ ($\Pi_i{}^0$ and $\bar{\mathcal{K}}_i$ in our notation) were the only FC constraints as claimed in [5], we would not be able to construct the complete Poincaré gauge generator, but only its Lorentz piece. Consequently, $N_c = 3$ is not the correct result.

As we mentioned in section 5, Carlip treated $(b^i{}_0, \omega^i{}_0, \lambda^i{}_0)$ as Lagrange multipliers, and he worked in the phase space equivalent to our R_2 [6]. After identifying the secondary constraints (as defined in table 2), he relied on the Lagrangian formalism to justify the introduction of an *extra* constraint Δ (in section 4). Adding a constraint in this way is a serious step, which might influence dynamical content of the original theory. To prevent that, one needs a consistency control of the procedure which guarantees that the constraint content of the theory remains unchanged with respect to the genuine canonical treatment. In particular, one should clarify whether there exist some other Lagrangian expressions, beside Δ, that should be also treated as constraints. We have not found a satisfying analysis of these issues in [6]. The extra constraint Δ essentially coincides with our $\theta_{\alpha\beta}$.

For negative Λ, one can define the chiral version of TMG_Λ by demanding that one of the two central charges vanishes, $\mu\ell \mp 1 = 0$. Li et al. [9] argued that, while TMG_Λ for generic μ is unstable, the chiral version of the theory might be consistent. Grumiller et al. [7] studied the case $\mu\ell = 1$ in a reduced phase space formalism, which is simmilar to (but not identical with) the one used by Carlip. They found $N_c = 1$, but again, only after imposing the additional condition $\theta_{\alpha\beta} \approx 0$, whose canonical status was not discussed. Our results imply that transition to the chiral coupling does not have a critical influence on the form of the PB algebra. Hence, we have $N_c = 1$ also for the chiral coupling.

(2) As a consistency check of our analysis of constraints, we used the PB algebra to construct the canonical generator of Poincaré gauge transformations. The form of this generator is improved by adding suitable surface terms, and used to examine the AdS asymptotic structure of TMG_Λ. The result of this analysis leads to a remarkable conclusion: the conserved charges and asymptotic symmetries of TMG_Λ are the same as in 3D gravity with torsion, in the limit of vanishing torsion. It is interesting to note that we have

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here two theories with substantially different local properties (3D gravity with torsion is a topological theory, while TMG_Λ has one propagating degree of freedom), but still, they have classically identical asymptotic structures.

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A The algebra of constraints

In this appendix, we display the most important PBs that facilitate the evaluation of the consistency requirements. Starting from the basic relations $\{b^i{}_\mu, \pi_j{}^\nu\} = \delta^i_j \delta^\nu_\mu \delta(\boldsymbol{x} - \boldsymbol{x'})$ etc., we find the PBs between the primary constraints,

$$\begin{aligned}
&\{\phi_i{}^\alpha, \Phi_j{}^\beta\} = -2a\varepsilon^{0\alpha\beta}\eta_{ij}\delta\,, && \{\phi_i{}^\alpha, p_j^\beta\} = -\varepsilon^{0\alpha\beta}\eta_{ij}\delta\,,\\
&\{\Phi_i{}^\alpha, \Phi_j{}^\beta\} = -2a\mu^{-1}\varepsilon^{0\alpha\beta}\eta_{ij}\delta\,,
\end{aligned}$$

between the primary and secondary constraints,

$$\begin{aligned}
&\{\phi_i{}^\alpha, \bar{\mathcal{H}}_j\} = 2\Lambda\varepsilon_{ijk}p^{k\alpha}\delta\,, && \{\phi_i{}^\alpha, \bar{\mathcal{K}}_j\} = -\varepsilon_{ijk}\phi^{k\alpha}\delta\,,\\
&\{\phi_i{}^\alpha, \bar{\mathcal{T}}_j\} = \frac{\mu}{2a}\varepsilon_{ijk}\left(-\Phi^{k\alpha} + 2ap^{k\alpha}\right)\delta\,,\\
&\{\Phi_i{}^\alpha, \bar{\mathcal{H}}_j\} = -\varepsilon_{ijk}\phi^{k\alpha}\,, && \{\Phi_i{}^\alpha, \bar{\mathcal{K}}_j\} = -\varepsilon_{ijk}\Phi^{k\alpha}\delta\,,\\
&\{\Phi_i{}^\alpha, \bar{\mathcal{T}}_j\} = -\varepsilon_{ijk}p^{k\alpha}\delta\,,\\
&\{p_i{}^\alpha, \bar{\mathcal{H}}_j\} = \frac{\mu}{2a}\varepsilon_{ijk}\left(-\Phi^{k\alpha} + 2ap^{k\alpha}\right)\delta\,, && \{p_i{}^\alpha, \bar{\mathcal{K}}_j\} = -\varepsilon_{ijk}p^{k\alpha}\delta\,,
\end{aligned}$$

and the PBs between the secondary constraints,

$$\begin{aligned}
\{\bar{\mathcal{H}}_i, \bar{\mathcal{H}}_j\} &= 2\Lambda\varepsilon_{ijk}\bar{\mathcal{T}}^k\delta + \frac{\mu}{2a}\varepsilon^{0\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta}\delta\\
&\quad + \frac{\mu}{2a}\varepsilon_{ijk}\varepsilon_{mn}{}^k\lambda^m{}_\beta\left[2a\left(2ap^{n\beta} - \Phi^{n\beta}\right) + \phi^{n\beta}\right]\delta\,,\\
\{\bar{\mathcal{H}}_i, \bar{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\bar{\mathcal{H}}^k\delta\,,\\
\{\bar{\mathcal{H}}_i, \bar{\mathcal{T}}_j\} &= \frac{\mu}{2a}\varepsilon_{ijk}\left(-\bar{\mathcal{K}}^k + 2a\bar{\mathcal{T}}^k\right)\delta - \frac{\mu}{2a}\varepsilon^{0\alpha\beta}(\eta_{ij}\lambda_{\alpha\beta} + \lambda_{i\alpha}b_{j\beta})\delta\,,\\
&\quad - \frac{\mu}{2a}\varepsilon_{imk}\varepsilon_{jn}{}^k b^m{}_\beta\left[2a\left(2ap^{n\beta} - \Phi^{n\beta}\right) + \phi^{n\beta}\right] + \frac{\mu}{2a}\varepsilon_{imk}\varepsilon_{jn}{}^k p^{m\beta}\lambda^n{}_\beta\delta\,,\\
\{\bar{\mathcal{K}}_i, \bar{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\bar{\mathcal{K}}^k\delta\,, \qquad \{\bar{\mathcal{K}}_i, \bar{\mathcal{T}}_j\} = -\varepsilon_{ijk}\bar{\mathcal{T}}^k\delta\,,\\
\{\bar{\mathcal{T}}_i, \bar{\mathcal{T}}_j\} &= \frac{\mu}{2a}\varepsilon^{0\alpha\beta}b_{i\alpha}b_{j\beta}\delta + \frac{\mu}{2a}\left(b_{i\beta}p_j{}^\beta - b_{j\beta}p_i{}^\beta\right)\delta\,.
\end{aligned}$$

Next, we calculate the PBs between $(\theta_{0\beta}, \theta_{\alpha\beta})$ and the secondary constraints:

$$\begin{aligned}
\{\theta_{0\beta}, \bar{\mathcal{H}}_i\} &= \nabla'_\beta(\lambda_{i0}\delta) + \varepsilon_{imk}\Big(2\Lambda b^m{}_\beta + \mu\lambda^m{}_\beta\Big)b^k{}_0\delta\,,\\
\{\theta_{0\beta}, \bar{\mathcal{K}}_i\} &= -\varepsilon_{imk}\big(\lambda^m{}_\beta b^k{}_0 + \lambda^m{}_0 b^k{}_\beta\big)\delta\,,
\end{aligned}$$

$$
\begin{aligned}
\{\theta_{0\beta}, \bar{\mathcal{T}}_i\} &= -\nabla'_\beta(b_{i0}\delta) + \mu\varepsilon_{ijk}b^j{}_\beta b^k{}_0\delta\,,\\
\{\theta_{\alpha\beta}, \bar{\mathcal{H}}_i\} &= -\nabla'_\alpha(\lambda_{i\beta}\delta) - \varepsilon_{ijk}b^j{}_\alpha(2\Lambda b^k{}_\beta + \mu\lambda^k{}_\beta)\delta - (\alpha \leftrightarrow \beta)\,,\\
\{\theta_{\alpha\beta}, \bar{\mathcal{K}}_i\} &= 0\\
\{\theta_{\alpha\beta}, \bar{\mathcal{T}}_i\} &= -\nabla'_\beta(b_{i\alpha}\delta) - \mu\varepsilon_{ijk}b^j{}_\alpha b^k{}_\beta - (\alpha \leftrightarrow \beta)\,,
\end{aligned}
$$

and between Ψ and the secondary constraints:

$$
\begin{aligned}
\{\Psi, \bar{\mathcal{H}}_i\} &= \mu\left[\nabla'_\beta(\lambda^\beta{}_i\delta) - \varepsilon_{ijk}h^{j\alpha}(2\Lambda b^k{}_\alpha + \mu\lambda^k{}_\alpha)\delta\right]\,,\\
\{\Psi, \bar{\mathcal{K}}_i\} &= -\mu\varepsilon_{ijk}\left(h^{j0}\lambda^k{}_0 + b^j{}_0\lambda^0{}_k\right)\delta\,,\\
\{\Psi, \bar{\mathcal{T}}_i\} &= \mu\left[-\nabla'_\beta(h_i{}^\beta\delta) + \mu\varepsilon_{ijk}h^{j0}b^k{}_0\delta\right]\,.
\end{aligned}
$$

Finally, we display the PBs among the secondary first class constraints $(\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_j)$:

$$
\begin{aligned}
\{\hat{\mathcal{H}}_i, \hat{\mathcal{H}}_j\} &= -\frac{\mu}{2a}\varepsilon_{ijk}\lambda^k{}_m\hat{\mathcal{K}}^m\delta\,,\\
\{\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\hat{\mathcal{H}}^k\delta\,,\\
\{\hat{\mathcal{K}}_i, \hat{\mathcal{K}}_j\} &= -\varepsilon_{ijk}\hat{\mathcal{K}}^k\delta\,.
\end{aligned}
$$

B Dirac brackets

The phase space R_1 is defined by the second class constraints $X_A := (\phi_i{}^\alpha, \Phi_i{}^\alpha, p_i{}^\alpha)$. To construct the corresponding DBs, we consider the 18×18 matrix Δ_1 with matrix elements $(\Delta_1)_{AB} = \{X_A.X_B\}$:

$$
\Delta_1 = \begin{pmatrix}
\{\phi_i{}^\alpha, \phi_j{}^\beta\} & \{\phi_i{}^\alpha, \Phi_j{}^\beta\} & \{\phi_i{}^\alpha, p_j{}^\beta\}\\
\{\Phi_i{}^\alpha, \phi_j{}^\beta\} & \{\Phi_i{}^\alpha, \Phi_j{}^\beta\} & \{\Phi_i{}^\alpha, p_j{}^\beta\}\\
\{p_i{}^\alpha, \phi_j{}^\beta\} & \{p_i{}^\alpha, \Phi_j{}^\beta\} & \{p_i{}^\alpha, p_j{}^\beta\}
\end{pmatrix}\,.
$$

The explicit form of Δ_1 reads:

$$
\Delta_1(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix}
0 & -2a & -1\\
-2a & -2a\mu^{-1} & 0\\
-1 & 0 & 0
\end{pmatrix} \otimes \varepsilon^{0\alpha\beta}\eta_{ij}\delta(\boldsymbol{x}, \boldsymbol{y})\,.
$$

The matrix Δ_1 is regular, and its inverse has the form

$$
\Delta_1^{-1}(\boldsymbol{y}, \boldsymbol{z}) = \frac{\mu}{a}\begin{pmatrix}
0 & 0 & a\mu^{-1}\\
0 & \frac{1}{2} & -a\\
a\mu^{-1} & a & 2a^2
\end{pmatrix} \otimes \varepsilon_{0\beta\gamma}\eta^{jk}\delta(\boldsymbol{y}, \boldsymbol{z})\,.
$$

The matrix Δ_1^{-1} defines the DBs in the phase space R_1:

$$
\{\phi, \psi\}_1^* = \{\phi, \psi\} - \{\phi, X_A\}(\Delta_1^{-1})^{AB}\{X_B, \psi\}\,.
$$

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The main part of the result is displayed in (5.1), while the remaining non-trivial first-level DBs involving $\lambda^i{}_0, p^{\beta 0}$ and $p_0{}^0$ are:

$$\{\lambda^i{}_\alpha, p^{\beta 0}\}_1^* = -\varepsilon_{0\alpha\gamma} h^{i\beta} p^{\gamma 0} \delta\,,$$
$$\{\lambda^i{}_0, p^{\beta 0}\}_1^* = h^{i\beta}\delta\,, \qquad\qquad \{\lambda^i{}_0, p_0{}^0\}_1^* = b^i{}_0\delta\,.$$

The reduced phase space R_2 is obtained from R_1 by imposing the additional second class constraints $Y_A := (\theta_{0\beta}, \Psi, p^{\alpha 0}, p_0{}^0)$. The corresponding 6×6 matrix Δ_2 reads:

$$\Delta_2 := \begin{pmatrix} \{\theta_{0\alpha}, \theta_{0\beta}\}_1^* & \{\theta_{0\alpha}, \Psi\}_1^* & \{\theta_{0\alpha}, p^{\beta 0}\}_1^* & \{\theta_{0\alpha}, p_0{}^0\}^* \\ \{\Psi, \theta_{0\beta}\}_1^* & \{\Psi, \Psi\}_1^* & \{\Psi, p^{\beta 0}\}_1^* & \{\Psi, p_0{}^0\}_1^* \\ \{p^{\alpha 0}, \theta_{0\beta}\}_1^* & \{p^{\alpha 0}, \Psi\}_1^* & \{p^{\alpha 0}, p^{\beta 0}\}_1^* & \{p^{\alpha 0}, p_0{}^0\}_1^* \\ \{p_0{}^0, \theta_{0\beta}\}_1^* & \{p_0{}^0, \Psi\}_1^* & \{p_0{}^0, p^{\beta 0}\}_1^* & \{p_0{}^0, p_0{}^{0*}\} \end{pmatrix}.$$

The explicit form of Δ_2 is:

$$\Delta_2(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} B & A \\ -A^T & 0 \end{pmatrix} \delta(\boldsymbol{x} - \boldsymbol{y})\,,$$

where

$$A := -\begin{pmatrix} \delta^\beta_\alpha & g_{0\alpha} \\ -g^{0\beta} & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 2\varepsilon_{0\alpha\beta}\left[a\mu g_{00} - \lambda_{00}\right] & \delta^0_\alpha \\ \delta^0_\beta & 0 \end{pmatrix}.$$

The inverse of Δ_2 is given by

$$(\Delta_2)^{-1}(\boldsymbol{y}, \boldsymbol{z}) = \begin{pmatrix} 0 & -(A^T)^{-1} \\ A^{-1} & A^{-1}B(A^T)^{-1} \end{pmatrix} \delta(\boldsymbol{y} - \boldsymbol{z})\,,$$
$$A^{-1} = \frac{1}{g_{00}g^{00}} \begin{pmatrix} -\delta^\beta_\alpha + \varepsilon_{0\alpha\gamma}\varepsilon^{0\beta\epsilon} g^{0\gamma} g_{0\epsilon} & -g_{0\alpha} \\ g^{0\beta} & 1 \end{pmatrix}.$$

The DBs in R_2 are the same as those in (5.1).

C 3D gravity with torsion in brief

Here, we give here a short review of some relevant features of the topological Mielke-Baekler model [10, 11]. The model is defined by the Lagrangian

$$L = 2ab^i R_i - \frac{\Lambda}{3}\varepsilon_{ijk} b^i b^j b^k \; + \alpha_3 L_{\rm CS}(\omega) + \alpha_4 b^i T_i\,.$$

In the non-degenerate sector with $\alpha_3\alpha_4 - a^2 \neq 0$, the gravitational field equations have the form

$$2T_i = p\varepsilon_{ijk} b^j b^k\,, \qquad\qquad 2R_i = q\varepsilon_{ijk} b^j b^k\,,$$

where

$$p := \frac{\alpha_3\Lambda + \alpha_4 a}{\alpha_3\alpha_4 - a^2}\,, \qquad\qquad q := -\frac{(\alpha_4)^2 + a\Lambda}{\alpha_3\alpha_4 - a^2}\,.$$

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The Riemannian piece of the Cartan curvature reads:

$$2\tilde{R}_i = \Lambda_{\text{eff}}\varepsilon_{ijk}b^j b^k\,, \qquad\qquad \Lambda_{\text{eff}} := q - \frac{p^2}{4}\,.$$

In the AdS sector of the theory, where the effective cosmological constant Λ_{eff} is negative, we have $\Lambda_{\text{eff}} =: -1/\ell^2$.

The surface term of the improved canonical generator reads:

$$\begin{aligned}
\Gamma &:= -\int_0^{2\pi} d\varphi\,\left(\xi^0\mathcal{E}^1 + \xi^2\mathcal{M}^1\right)\,, \qquad\qquad (C.1)\\
\mathcal{E}^\alpha &= 2\varepsilon^{0\alpha\beta}\left[\left(a + \frac{\alpha_3 p}{2}\right)\omega^0{}_\beta + \left(\alpha_4 + \frac{ap}{2}\right)b^0{}_\beta + \frac{a}{\ell}b^2{}_\beta + \frac{\alpha_3}{\ell}\omega^2{}_\beta\right]b^0{}_0\,,\\
\mathcal{M}^\alpha &= -2\varepsilon^{0\alpha\beta}\left[\left(a + \frac{\alpha_3 p}{2}\right)\omega^2{}_\beta + \left(\alpha_4 + \frac{ap}{2}\right)b^2{}_\beta + \frac{a}{\ell}b^0{}_\beta + \frac{\alpha_3}{\ell}\omega^0{}_\beta\right]b^2{}_2\,.
\end{aligned}$$

The values of the surface term for $\xi^0 = 1$ and $\xi^2 = 1$ define the energy and angular momentum of the system, respectively. In particular, the conserved charges for the BTZ black hole read:

$$E = m + \frac{\alpha_3}{a}\left(\frac{pm}{2} - \frac{J}{\ell^2}\right)\,, \qquad\qquad M = J + \frac{\alpha_3}{a}\left(\frac{pJ}{2} - m\right)\,. \qquad (C.2)$$

The canonical algebra of the improved generators is characterized by two different central charges:

$$c^\mp = \frac{3\ell}{2G} + 24\pi\alpha_3\left(\frac{p\ell}{2} \pm 1\right)\,. \qquad (C.3)$$

According to the field equations, the vanishing of torsion can be described by three equivalent conditions:

$$p = 0\,, \qquad\qquad q = \frac{\Lambda}{a} = -\frac{1}{\ell^2}\,, \qquad\qquad \alpha_4 = \frac{\alpha_3}{\ell^2}\,. \qquad (C.4)$$

This case is of particular interest for comparison with TMG_Λ. Note that the Chern-Simons coupling constant μ in TMG_Λ is related to α_3 by $\alpha_3 = a/\mu$.

For $p = 0$, the treatment of the chiral limit of 3D gravity with torsion demands an extension of the canonical analysis to the sector $\alpha_3\alpha_4 - a^2 = 0$.

References

[1] S. Deser, R. Jackiw and G. 't Hooft, *Three-dimensional Einstein gravity: dynamics of flat space*, *Annals Phys.* **152** (1984) 220 [SPIRES];
S. Deser and R. Jackiw, *Three-dimensional cosmological gravity: dynamics of constant curvature*, *Annals Phys.* **153** (1984) 405 [SPIRES];
E.J. Martinec, *Soluble systems in quantum gravity*, *Phys. Rev.* **D 30** (1984) 1198 [SPIRES].

[2] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories*, *Phys. Rev. Lett.* **48** (1982) 975 [SPIRES]; *Topologically massive gauge theories*, *Ann. Phys.* **140** (1982) 372 [*Erratum ibid.* **185** (1988) 406] [*Annals Phys.* **281** (2000) 409] [SPIRES].

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[3] For a review of the subject and an extensive list of references, see: S. Carlip, *Conformal field theory, $(2+1)$-dimensional gravity, and the BTZ black hole*, *Class. Quant. Grav.* **22** (2005) R85 [gr-qc/0503022] [SPIRES].

[4] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [hep-th/0508218] [SPIRES];
S.N. Solodukhin, *Holography with gravitational Chern-Simons term*, *Phys. Rev.* **D 74** (2006) 024015 [hep-th/0509148] [SPIRES];
K. Hotta, Y. Hyakutake, T. Kubota and H. Tanida, *Brown-Henneaux's canonical approach to topologically massive gravity*, *JHEP* **07** (2008) 066 [arXiv:0805.2005] [SPIRES].

[5] M.-i. Park, *Constraint dynamics and gravitons in three dimensions*, *JHEP* **09** (2008) 084 [arXiv:0805.4328] [SPIRES].

[6] S. Carlip, *The constraint algebra of topologically massive AdS gravity*, *JHEP* **10** (2008) 078 [arXiv:0807.4152] [SPIRES].

[7] D. Grumiller, R. Jackiw and N. Johansson, *Canonical analysis of cosmological topologically massive gravity at the chiral point*, contribution to Wolfgang Kummer Memorial Volume, arXiv:0806.4185 [SPIRES].

[8] S. Deser and X. Xiang, *Canonical formulations of full nonlinear topologically massive gravity*, *Phys. Lett.* **B 263** (1991) 39 [SPIRES];
I.L. Buchbinder, S.L. Lyahovich and V.A. Krychtin, *Canonical quantization of topologically massive gravity*, *Class. Quant. Grav.* **10** (1993) 2083 [SPIRES].

[9] W. Li, W. Song and A. Strominger, *Chiral gravity in three dimensions*, *JHEP* **04** (2008) 082 [arXiv:0801.4566] [SPIRES];
A. Strominger, *A Simple proof of the chiral gravity conjecture*, arXiv:0808.0506 [SPIRES].

[10] E.W. Mielke and P. Baekler, *Topological gauge model of gravity with torsion*, *Phys. Lett.* **A 156** (1991) 399 [SPIRES];
P. Baekler, E.W. Mielke and F.W. Hehl, *Dynamical symmetries in topological $3-D$ gravity with torsion*, *Nuovo Cim.* **B 107** (1992) 91 [SPIRES].

[11] M. Blagojevic and B. Cvetkovic, *Canonical structure of $3D$ gravity with torsion*, in *Progress in general relativity and quantum cosmology, Vol. 2*, Ch. Benton ed., Nova Science Publishers, New York U.S.A. (2006), pg. 103–123, gr-qc/0412134 [SPIRES]; *Asymptotic charges in 3d gravity with torsion*, *J. Phys. Conf. Ser.* **33** (2006) 248 [gr-qc/0511162] [SPIRES].

[12] P.A.M. Dirac, *Lectures on quantum mechanics*, Belfer Graduate School of Science, Yeshiva University, New York U.S.A. (1964).

[13] M. Blagojević, *Gravitation and gauge symmetries*, IoP Publishing, Bristol U.K. (2002) [SPIRES];
T. Ortín, *Gravity and strings*, Cambridge University Press, Cambridge U.K. (2004) [SPIRES].

[14] M. Banados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [hep-th/9204099] [SPIRES];
M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the $(2+1)$ black hole*, *Phys. Rev.* **D 48** (1993) 1506 [gr-qc/9302012] [SPIRES].

[15] L. Castellani, *Symmetries in constrained Hamiltonian systems*, *Annals Phys.* **143** (1982) 357 [SPIRES].

JHEP05(2009)073

[16] T. Regge and C. Teitelboim, *Role of surface integrals in the Hamiltonian formulation of general relativity*, *Annals Phys.* **88** (1974) 286 [SPIRES].

[17] J.D. Brown and M. Henneaux, *On the poisson brackets of differentiable generators in classical field theory*, *J. Math. Phys.* **27** (1986) 489 [SPIRES].

[18] M. Blagojevic and B. Cvetkovic, *Black hole entropy from the boundary conformal structure in 3D gravity with torsion*, *JHEP* **10** (2006) 005 [gr-qc/0606086] [SPIRES].

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